The Axiom of Choice is a Logical Truth

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The Axiom of Choice is a set theoretic principle which says

Definition.

Choice: For any set S of non-empty sets, there is a function f where, for every $T \in S$, $f(T) \in T$.

It is independent of the axioms of Zermelo-Frankel set theory, and it is sometimes taken to be controversial. Some of the controversy stems from the fact that it can be used to prove some surprising results — notably the well-ordering theorem (which says that every set can be well-ordered) and the Banach-Tarski theorem.

But, from the perspective of informal logic, there is something somewhat troubling about the controversy. Consider the following:

Definition.

Finite Choice: For any finite set S of non-empty sets, there is a function f where, for every $T \in S$, $f(T) \in T$.

Finite Choice *is* a theorem of Zermelo-Frankel set theory, or ZF. And you prove it by use of the very simple proof techniques of existential generalization and existential instantiation. As used in informal proofs, when we know that there is an F, these let us 'name' or 'pick' an arbitrary F, show that it is a G, and then conclude that something is a G. We can use these techniques to prove finite choice: For each non-empty $T \in S$, we 'pick' an arbitrary object $a_T \in T$, and then we define $f(T) = a_T$ for each of these objects. Thus defined, f *is* a finite choice function for S, and so we conclude Finite Choice itself by existential generalization.

When first encountering set theory, naive students often want to extend this practice infinitely. 'We know that each set is non-empty', they say, 'so for each $T \in S$, there is an $a_T \in T$. If we define $f(T) = a_T$, we have our choice function. So by existential generalization, there is a choice function for S.'

These naive students' more sophisticated instructors then scold them for their mistake: 'Since each $T \in S$ is non-empty, you can conclude that there is something in each such T; but you can't go giving it a name like a_T or putting it into a function, because that is to assume the axiom of choice.'

But is the sophisticated instructor's objection anything more than mathematical prejudice? The techniques used in proving finite and infinite choice both *seem* natural; the moves feel, intuitively, like they ought to be right, and keep feeling that way until enough sophisticated instructors beat our naive instincts out of us. Furthermore, the analogy between the finite case and the infinite case seems so close that the division looks, quite frankly, arbitrary.

Against the arbitrariness charge the sophisticated instructor has this to say: 'True, the way we worded both "proofs" looked similar. But there is a very crucial difference between the two. In the first case, it is in principle possible to 'pick' the a_T's one at a time, by going through the sets one by one. Then what looked like a "global" picking of many a_T's can be seen as really an iterated process of picking one a_T, and then another, and then another. That can be done in the finite case, but not in the infinite case, as we would always have another a_T to pick.'

Before evaluating the sophisticated instructor's response, note that the debate has drifted away from set theory. The question is no longer one about what sets there are, but what kind of logical principles are allowed in informal proofs. The naive student thinks a procedure like the following should be legit:

> **Generalized Existential Instantiation:** If there are some Fs which each R some Gs, then we can pick an arbitrary G_F for each F to reason about in the process of deploying existential instantiation.

(What do we mean by 'in the process of deploying existential instantiation'? Just that, when we're done with the reasoning, we shouldn't be making claims about any particular G_F (since they were arbitrary) but something that has generalized away from them.) The sophisticated instructor rejects existential instantiation, and only allows a finite version of it because it can, in some sense, be 'reduced' to

> **Single-Case Existential Instantiation:** If an F Rs some Gs, then we can pick an arbitrary G_F to reason about in the process of deploying existential instantiation.

Sets only get involved because, in set-theoretic contexts, if we *want* to deploy Generalized Existential Instantiation but aren't allowed to on purely logical grounds, the function guaranteed by the Axiom of Choice helps make up the difference.

To return to the sophisticated instructor's response: How plausible is the reduction of the Finite Generalized principle to the Single-Case principle? Note that, in many instances, the reduction itself could never be carried out. Let S be a set with more non-empty sets than there are particles in the observable universe. Clearly nobody will ever manage to prove the existence of a choice function for S by repeated application of Single-Case Instantiation. So why is it so important that this could 'in principle' be done?

Sometimes the sophisticated instructor tells the naive student something like, "You are assuming that there is some way to choose a G_F for each F. But without the axiom of choice you have no way to make all of these infinitely many choices." But this is a very strange thing to say. When we 'choose' a G_F, even in Single-Case Instantiation, we aren't really *choosing* anything. We're not reaching into a bag of Gs and pulling one out. Nor are we introducing some method for selecting some G for each F. We're introducing a name which is meant to refer to an arbitrary G that is R-ed by an associated F. That is: We're introducing a name which is to refer to some G that is R-ed by an F, and we don't particularly care *which* G is referred to by the name. But if we can do that for arbitrarily many finite Gs, then what stops us from introducing infinitely many such names? True, we could never write all infinitely many of them down; but we could never write down Graham's number of names, either.

A more sophisticated response the instructor might give would appeal to the 'G_F' notation. 'When you write down "G_F", what you're really writing down is "the result of applying some G-delivering function to F",' he says. 'In other words, "G_F" is functional notation, no different in principle than "$f(F)$". So to assume that each name "G_F" is well-defined is to assume the Axiom of Choice.'

If 'G_F' notation really is functional notation, the sophisticated instructor is right. And, of course, *if* there is a function f delivering a G for each F, we could use 'G_F' as another way of writing '$f(F)$'. But it seems strained to think that Generalized Instantiation *presupposes* the notion of a function. After all, even mathematical nominalists, who believe their are no functions whatsoever, can use Generalized Instantiation over finitely many Fs, no matter how large the number of Fs. When the nominalist says 'for each F, let G_F be one of the things that F R's', she doesn't contradict herself by a covert appeal to functions.

It seems, then, that the naive student was right after all. The sophisticated instructor's objections to Generalized Instantiation look ill-founded. If so, then Generalized Instantiation is a legitimate method of reasoning, quite independently of any concerns about sets.

A second line of reasoning supports this conclusion. Forget about proof techniques for a moment and think instead about plural quantification. Consider the following principle:

Definition.

Plural Choice: Suppose that there are some xxs which each R some yys. Then there are some zzs among the yys where every one of the xxs Rs exactly one of the zz.

It can take a moment or two to parse what this says, so here is another gloss:

> If there are some xxs and each one of the xxs has some representatives, then there are some representatives, the zzs, where each one of the xxs has exactly one representative among the zzs.

Once it's parsed, it seems hard to deny. Consider an instance of this: Suppose that there are some mothers which is each the mother of some daughters. Then there are some zzs where each mother is the mother of a unique daughter in the zzs. In fact there are many. Just look at the *zz*s and then squint away, for each mother, all daughters but one.

Surely this is true. The daughters are just *there*, whether or not we have some 'process' of squinting away their sisters. Their existence doesn't depend on any mental activity we can do; it's guaranteed by the fact that they are among the daughters had by the mothers in question. What would it take for Plural Choice to be false? The daughters are of course just there, hanging around; Plural Choice could only fail if somehow or other those daughters failed to 'form a collection' — that is, failed to be a value for the plural variable '*zz*'. But we are talking about *all* of the things, and so any things around are available as a plural variable for '*zz*', these daughters included.

Of course, I worded this in terms of 'mothers' and 'daughters', but there was nothing about familial relations necessary for the argument. And while there are may be only finitely many mothers and daughters in the actual world,[1] nothing in the argument made any use of that, either. No matter if the mothers outnumber the sets, if they each have some daughters, then there are some daughters where they each are the mother of exactly one of them.

We might have tried to make the same arguments about sets instead of plurals, of course. Those arguments would fail. But they would fail instructively. One of the lessons of the failure of naive set theory is that there can be some things of which there is no set of them. The sets that are not self-members, for instance. So we can't just say 'Clearly the things are there, so the set of them is there, too!' But plurals aren't like this. If there are some things, then. . . , well, there are some things. There is no 'gap' between the things being there and them being there, the way there is a gap between the things being there and them forming the set. This is why our naive reaction to Plural Choice is on sounder epistemic footing than our naive reaction to Choice.

So Plural Choice is true, I claim. In fact, I claim something further: It is *logically* true. If logical truth is truth in all (appropriately defined) models, then we can argue for this. A proper model theory for plural quantification ought to use plural quantifiers in the model theory itself, after the manner of Boolos 1985 and Rayo and Uzquiano 1999. In this case, the truth of Plural Choice is enough to guarantee the its truth in all models on such a model theory. So if logical truth is identified as truth of all models (of the right kind), Plural Choice is a logical truth.

Perhaps a truth-in-all-models understanding of logical truth is to be rejected. If so, I can only offer this in reply: Plural Choice seems to not only be true, but to be necessarily true in the distinctive sort of way of logical truths. Trying to imagine the failure of Plural Choice is not unlike trying to imagine the failure of existential generalization. Each mother has some daughters. How could there not be some daughters where each mother has exactly one among them? Where did those daughters go?

But, in the context of plural ZF set theory, Plural Choice entails the Axiom of

[1]Perhaps not, if the universe is infinitely large.

Choice. The reason is that in plural ZF, we have the principle that if S is any set each one of the yys are in S, then there is a set containing exactly the yys. So we reason:

Proof.

Let S be a set of non-empty sets. Let the xxs be the members of S. Note that each S has as a member some yys. Then by Plural Choice there are some zzs among the yys where each one of the xxs has exactly one element in the zzs. But the yys form a set — namely, the set $\bigcup S$ — so the zzs form a set as well. Call it Z. Then we can define the Choice Function f for S by setting $f(T) = Z \cap T$. $\therefore$

So Plural Choice is true, and — say I — is a logical truth. But it, against the background of ZF, entails the Axiom Choice. So the Axiom of Choice is in fact a consequence of (plural) ZF, not requiring an extra axiom, and once we recognize the truths of plural logic as such we can recognize this fact.

What of Generalized Existential Instantiation? It should be clear that the truth of Plural Choice licenses it as well. Suppose there are some Fs which each R some Gs; then by Plural Choice, there are some zzs among the Gs, where each F Rs exactly one of them. We can then use Single-Case Existential Instantiation applied to *plural* quantification to choose some arbitrary aas to be the aas, and then define, for each x among the Fs, G_x to be the unique thing among the aas that is R-ed by x. We can then reason about the G_xs; since their definition was in terms of the aas, to deploy existential instantiation we'll have to generalize away the aas and anything defined in terms of them, including each term G_x. Thus Generalized Instantiation reduces, in the presence of Plural Choice, to Plural One-Case Instantiation.

Conversely, it is not difficult to prove Plural Choice from Generalized Instantiation. Suppose there are some xxs which each R some yys; by Generalized Existential Instantiation, we can say, for each x among the xxs, let y_x be one of the yys that x R's. Let the aas consist of all and only the y_xs; then clearly the aas are among the yys and each one of the xxs R-s exactly one of them Existentially Generalizing on the aas, there are some zzs that are among the yys, where each of the xxs R's exactly one of them.

The point, of course, is not to argue circularly, but rather to show that the two principles are intertwined and the support each of them receives is mutual. It seems, *prima facie*, that Generalized Instantiation ought to be true; the biggest resistance to it comes from the sophisticated instructors who have been long-trained to suspect it as covertly smuggling in an appeal to Choice. Plural Choice likewise seems independently true, and the main reason for doubting our naive reactions in the set-theoretic case — our acknowledgement that there can be some things without there being a *set* of those things — does not apply. Given that either can be used to support the other, there seems to be an excellent case that Plural Choice is a logical truth, Generalized Existential Instantiation a valid informal proof procedure, and Choice a logical consequence of plural ZF. Resistance to the Axiom of Choice is misguided, and ought to be done away with.

References

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