The Expressive Equivalence of Plural and Second-Order Logic

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In the wake of George Boolos's [1, 2] forceful defense of plural quantification, philosophical orthodoxy has embraced the legitimacy of plural logic. This isn't to say there hasn't been resistance [7, 5, 9], but the tide has largely turned. Orthodoxy, however, hasn't been so kind to second-order logic generally. Despite pockets of acceptance [8, 10, 12], the general philosophical populace still seems to look at it askance.

Why is this? It could be a number of things. Perhaps Quine's old slogan about bound variables has something to do with it. But one objection has it that second-order logic isn't kosher because it's too bound up with set theory. That is to say, second-order logic isn't *set-theoretically absolute*: which of its formulas are valid depends on what background set-theoretic universe its model theory is given in.

There is, for instance, a second-order formula CH that is true on every model if and only if the continuum hypothesis is true. If the background set-theoretic universe is one in which the hypothesis holds, CH has no countermodels; otherwise, it does. But (goes the objection) this is bad. Perhaps it is bad because logic is supposed to be topic-neutral. Or perhaps it is bad because logical truths should at least be in principle *a priori* knowable to a strong enough reasoner, and even an ideal reasoner should be agnostic about the truth of CH. Or perhaps it is bad for some other reason. (See [11], section 4 for discussion.)

An orthodox acceptance of plural logic combined with a rejection of second-order logic motivated by these sorts of concerns is, however, unstable. It has long been part of the folklore about plural logic that, if supplemented with a theory of pairing, it can recover the full expressive power of second-order logic. ([2], n. 3 and [6], p. 71; cf. also [3] p. 150.) If that is right, then there ought to be a (satisfiable) theory of pairing P and a translation of CH, PCH, such that PCH is true on every (plural) model of P if and only if CH is true on every (second-order) model simpliciter. But if that's right, then $P \rightarrow PCH$ will be true on every plural model if and only if the continuum hypothesis is true. If that's bad news for second-order logic, it's bad news for plural logic, too.

Orthodoxy can tollens or ponens here, welcoming second-order logic or casting out the plural. I'm not going to take a stand on which. But the argument above relies on the folklore, and we might wonder: Is the folklore right? It certainly seems like it *ought* to be true. But given that the folklore would make plural logic no more set-theoretically absolute than second-order logic, we might fairly want to rely on more than just folklore. We might, in fact, want something like a *proof* of it. Unfortunately, I have been unable to find a published proof of this claim. So I offer one here.

1 Model Theory

The mutual interpretability of plural logic and monadic second-order logic — that is, second-order logic with only monadic predicate variables — is well-known. So it will suffice to prove that monadic second-order logic in the presence of a pairing function is expressively equivalent to full second-order logic.

A second-order language contains logical symbols — we will take them as '$\sim$', '$\wedge$', '$\exists$', and '$=$' —, a stock of names and n-adic predicates, and a countable infinity of first-order variables and of n-adic second-order variables for each n. (For simplicity we treat all functions as reduced to predicates in the usual way.) A **second-order model** is an ordered pair $\langle D, I\rangle$ of a non-empty domain and an interpretation function I, where if α is a name, $I(\alpha) \in D$, and if Π^n is an n-adic predicate, $I(\Pi^n) \subseteq D^n$.

If $\mathcal{M}$ is a model of a language $\mathscr{L}$, a variable assignment a is any function from the variables of $\mathscr{L}$ where, if x is a first-order variable, $a(x) \in D$, and if X^n is an n-adic predicate variable, $a(X^n) \subseteq D^n$. If a is a variable assignment and x any variable (of any order), $a[e/x]$ is the assignment where $a[e/x](y) = a(y)$ when $x \neq y$ and $a[e/x](x) = e$.

For any model $\mathcal{M}$, variable assignment a, and constant or variable σ, we define $[\![\sigma]\!]_{\mathcal{M},a}$ as $I(\sigma)$ if it is a constant or $a(\sigma)$ if it is a variable, and suppress the '$\mathcal{M}$' subscript when it is clear from context. Truth on a model relative to a variable assignment is defined as usual. We give the recursive clauses here for definiteness. To reduce clutter, we abbreviate sequences '$\tau_1, \ldots, \tau_n$' as '$\vec{\tau}$':

Definition.

If $\mathcal{M} = \langle D, I\rangle$ is a model and a a variable assignment:

(TAP) $\mathcal{M}, a \models \Pi^n\vec{\alpha}$ iff $\langle\overrightarrow{[\![\alpha]\!]_a}\rangle \in [\![\Pi^n]\!]_a$

(TA=) $\mathcal{M}, a \models \alpha = \beta$ iff $[\![\alpha]\!]_a = [\![\beta]\!]_a$

(T$\sim$) $\mathcal{M}, a \models \sim\phi$ iff $\mathcal{M}, a \not\models \phi$

(T$\wedge$) $\mathcal{M}, a \models \phi \wedge \psi$ iff $\mathcal{M}, a \models \phi$ and $\mathcal{M}, a \models \psi$

(T$\exists$) $\mathcal{M}, a \models \exists x\phi$ iff for some $o \in D$, $\mathcal{M}, a[o/x] \models \phi$

(T$\exists$2) $\mathcal{M}, a \models \exists X^n\phi$ iff for some $S \subseteq D^n$, $\mathcal{M}, a[S/X^n] \models \phi$

2 The Simple Theory of Pairing

If $\mathscr{L}$ is a second-order language, we let $\mathscr{L}^{¶}$ be the expansion of that language by the addition of a new triadic predicate ¶. If the theory to be given does its job right, '¶xyz' will say that $\langle x, y\rangle = z$. We have the following abbreviations.

Definition.

$$
\begin{aligned}
Pr(z) &:= \exists x \exists y \P xyz \\
x \prec_1 z &:= \exists y \P xyz \\
y \prec_2 z &:= \exists x \P xyz \\
y \prec_0 z &:= y \prec_1 z \vee y \prec_2 z \\
x \prec_i^* p &:= \forall X[(Xx \wedge \forall y \forall z[(Xy \wedge y \prec_i z) \to Xz]) \to Xp] \\
\downarrow(X,p) &:= Xp \wedge \forall y((Xy \wedge p \neq y) \to y \prec_0^* p) \\
\downarrow^*(X,p) &:= \downarrow(X,p) \wedge \forall x \forall y \forall z([Xx \wedge Xz \wedge x \prec_0^* y \wedge y \prec_0^* z] \to Xy) \\
Pth(X,p) &:= \downarrow^*(X,p) \wedge \forall x \forall y \forall z[(Xx \wedge Xy \wedge Xz \wedge x \prec_0 z \wedge y \prec_0 z) \to x = y]
\end{aligned}
$$

Intuitively, $Pr(z)$ says that z is a pair; $x \prec_1 z$ says that x is the first member of z, and $x \prec_2 z$ says x is z's second member. $\prec_0$ is the first-or-second member relation; and for any $\prec_i$, $\prec_i^*$ is its transitive closure.

The others are less straightforward. Roughly, $\downarrow(X,p)$ says that p is in X and everything else in X is a 'descendant' of p under the $\prec_0$ relation. $\downarrow^*(X,p)$ says all this plus, if two things are in X and there is a $\prec_0$-chain running between them, then every link of that chain is in X. These two are used together to define $Pth(X,p)$, which says that X is a non-branching 'path' of descendants running down from p. Notice that a path needn't be a *complete* path; if p_1 is in p_2 and p_2 is in p_3, then $X = \{p_3, p_2\}$ is a path from p_3 even though it doesn't contain p_1. Also notice that $\{x\}$ is a path from x, and if x is a non-pair, then it is indeed the only such path.

The axioms of our theory of pairing are:

(P1) $\forall x \forall y \exists! z(\P xyz)$

(P2) $\forall x_1 \forall x_2 \forall y_1 \forall y_2 \forall z([\P x_1 y_1 z \wedge \P x_2 y_2 z] \to [x_1 = x_2 \wedge y_1 = y_2])$

(P3) $\forall x {\sim} x \prec_0^* x$

(P4) $\forall p \forall X(Pth(X,p) \to \exists Y(Pth(Y,p) \wedge X \subseteq Y \wedge \exists y[{\sim}Pr(y) \wedge Yy)])$

(P1) tells us that pairing is functional, and (P2) tells us that a given pair has a unique decomposition into its first and second member. (P3) rules out 'loops' of membership, and (P4) tells us that every path of descendants of a pair p can be extended to a path that terminates in some non-pair. It's essentially a well-foundedness axiom that tells us that it can't be pairs all the way down. We use **P** for the theory consisting of (P1)–(P3) and P for the conjunction of **P**.

If $\mathcal{M}$ is a model and $o_1, o_2 \in D$, we write $o_1 \lessdot_i^{\mathcal{M}} o_2$ if o_1 and o_2 satisfy $\prec_i$, and write '$\lessdot^{\mathcal{M}}$' for '$\lessdot_0^{\mathcal{M}}$'. We drop the superscript when context makes it clear (which for our purposes is always), and use asterisks for $\lessdot_i$'s transitive closures. Note that, on a model of **P**, $\lessdot^*$ will be a strict partial order. Now we can prove:

Lemma 1.

If $\mathcal{M} \models \mathbf{P}$, then the function $rank_{\lessdot}$ is well-defined on D.

Proof.

Since $\lessdot^*$ is a partial order it suffices to show that $\lessdot$ is well-founded on D. First, let P be any $\lessdot$-path in D. Note that an element of D is $\lessdot$-minimal iff it satisfies $\sim P(z)$. Since $\mathcal{M} \models$ (PL4), P can be extended to a path P' that terminates in a $\lessdot$-minimal element. Since every point in the path is $\lessdot^*$-related to this element, the path must be finite. Now let X be an arbitrary subset of D. If X had no $\lessdot$-minimal element it would contain an infinite chain $x_1 \gtrdot^* x_2 \gtrdot^* \ldots$ which could be expanded to an infinite path, contrary to what we have just seen. $\therefore$

We want to verify that $\mathbf{P}$ really does count as a theory of *pairing*. To do that, we show that its truth on a model gives a pair-like structure to that model's domain. A definition will help with this.

Definition.

If $\mathcal{M} = \langle D, I \rangle$ is a model, then $\mathcal{M}$ is

pair-saturated iff for every $x, y \in D$, $\langle x, y \rangle \in D$;

pair-closed iff for every $\langle x, y \rangle \in D$, $x, y \in D$;

pair-well-founded iff there is no infinite chain $p_1, p_2, \ldots$ of pairs where each p_i is in the domain and has p_{i+1} as a (first or second) element; and

pair-faithful iff $I(\P) = \{\langle x, y, z \rangle : \langle x, y \rangle = z\}$

If $\mathcal{M}$ meets all four conditions, then it is **pair-normal**.

Note the following.

Lemma 2.

If $\mathcal{M}$ is pair-closed and pair-well-founded, then every pair is decomposable into a finite set of non-pairs.

Proof.

Let D be pair-closed and pair-well-founded and let p be a pair in P. Let $\in^*$ be the transitive closure of membership on pairs. By pair-well-foudnedness $X = \{y : y \in^* p\} \subseteq D$ contains no infinite chains. Thus every chain in D terminates; by pair-closure, any terminal node of such a chain is a non-pair. Furthermore, X is a binary tree under $\in$ and contains no infinite paths, so by König's lemma, X is finite. $\therefore$

Now we're ready to show:

Theorem 3.

$\mathcal{M}$ is a model of **P** iff $\mathcal{M}$ is isomorphic to some pair-normal model.

Proof.

For the right-to-left direction we show that, if a model $\mathcal{M}'$ is pair-normal, then it is a model of **P**. For (P1)–(P2) it is secured by the fact that ¶ expresses the pairing function on a pair-faithful model. In the case of (P4) this is secured by lemma 2; in (P3), by the fact that $\in^*$ (the transitive closure of the membership relation on pairs) is a partial order.

For the left-to-right direction, let $\mathcal{M} = \langle D, I \rangle$ be a model of **P**. Since it is a model of (P4), there must be some $o \in D$ where $\mathcal{M}, a[o/x] \models \sim Pr(x)$. Let N be the set of such $o \in D$, and let N' be a set of non-pairs and f_- a one-to-one correspondence between N and N'. Finally, let D' be the closure of N' under pairing; that is, let D' be the smallest (under $\subseteq$) set X such that $N' \subseteq X$ and $x, y \in X$ imply $\langle x, y \rangle \in X$. Now we extend f_- to a one-to-one correspondence between D and D' by:

$$f(o) = \begin{cases} f_-(o) \text{ if } o \in N, \text{ and otherwise} \\ \langle f(o_1), f(o_2) \rangle \text{ where } \mathcal{M}, a[o_1/x][o_2/y][o/z] \models \P xyz. \end{cases}$$

First, we show that f is well-defined as a function by induction on the $\lessdot$-rank of o. When $rank_\lessdot(o) = 0$, $o \in N$ and so $f(o) = f_-(o)$. If $rank_\lessdot(o) > 0$, then there must be some e where $e \lessdot o$, which means that $\mathcal{M}, a[e/x][o/y] \models \exists x \prec_1 y \vee x \prec_2 y$. Whichever it is, there must be some e' where either $\mathcal{M}, a[e/x][e'/y][o/z] \models \P xyz$ or $\mathcal{M}, a[e'/x][e/y][o/z] \models \P xyz$. Thus $e, e' \lessdot o$. Furthermore, since $\mathcal{M} \models$ (P2), this e and e' are unique. But they both have rank less than o's, so $f(e)$ and $f(e')$ are well-defined by the induction hypothesis. Setting e_1 and e_2 to the respective o_1 and o_2 as appropriate, $\langle f(o_1), f(o_2) \rangle$ exists and is unique.

Next, to show that f is one-to-one. Suppose $f(o) = f(o')$. If $o \in D$ then $o = o'$ by f_-'s being one-to-one. Otherwise, $o = \langle f(o_1), f(o_1) \rangle = o'$, where $\mathcal{M}, a[o_1/x][o_2/y][o/z] \models \P xyz$ and $\mathcal{M}, a[o_1/x][o_2/y][o'/z] \models \P xyz$. But since $\mathcal{M} \models$ (PL2), $o = o'$ (by the uniqueness clause).

Finally, that f is onto D'. Let $o \in D'$. By lemma 2, $rank_\in$ is finite on D'. Proof by induction on $rank_\in$. When $rank_\in(o) = 0$, then $f^{-1}(o) = f_-^{-1}(o)$ since f_- is a one-to-one correspondence. If $rank_\in(o) = n > 0$, then $o = \langle o_1, o_2 \rangle$ by construction of D', with $rank_\in(o_1)$, $rank_\in(o_2) < n$. By the induction hypothesis, $f^{-1}(o_1)$ and $f^{-1}(o_2)$ exist. Since $\mathcal{M} \models$ (PL1), there is a unique $e \in D$ where $\mathcal{M}, a[f^{-1}(o_1)/x][f^{-1}(o_2)/y][e/z] \models \P xyz$, so $f(e) = \langle f^{-1}(f(o_1)), \langle f^{-1}(f(o_2)) \rangle = \langle o_1, o_2 \rangle = o$.

Finally we construct a model $\mathcal{M}' = \langle D', I' \rangle$ where $I'(\sigma)$ is the image of $I(\sigma)$ under f. It is clear that $\mathcal{M}'$ is then isomorphic to $\mathcal{M}$ under f. Given the isomorphism, $I'(\P) = \{\langle x, y, z \rangle : \langle f^{-1}(x), f^{-1}(y), f^{-1}(z) \rangle \in I(\P)\}$; but by our construction of f, $z = f(f^{-1}(z)) = \langle f^{-1}(x), f^{-1}(y) \rangle = \langle x, y \rangle$, so $I'(\P) = \{\langle x, y, z \rangle : \langle x, y \rangle = z\}$. So $\mathcal{M}$ is pair-faithful. ∴

3 Field-Conservativeness of P

We can use the lemmas of the previous section to show that **P** is **Field-conservative** [4]. More precisely, we define a restriction function $(\cdot)^R$ that restricts the quantifiers by:

$$(\exists x \phi)^R = \exists x (\sim Pr(x) \wedge \phi^R)$$
$$(\exists X^n \phi)^R = \exists X^n (\forall \vec{y} (X^n \vec{y} \rightarrow \bigwedge \sim Pr(y_i)) \wedge \phi^R)$$

and for any other type of formula ψ, $\psi^R = \psi$. Then we can show

Theorem 4.

Let Γ be a set of sentences that do not contain ¶. Then Γ has a model if and only if $\Gamma^R + \mathbf{P}$ has a model.

Proof.

Let $\mathcal{M}$ be any model of a ¶-free language and assume without loss of generality that $\mathcal{M}$'s domain contains no pairs. (Else we find an isomorphic model of this sort.) Let $\mathcal{M}^+$ be the result of closing $\mathcal{M}$ under ordered pairs and interpreting ¶ so as to make $\mathcal{M}^+$ pair-faithful. Conversely, if $\mathcal{N} \models \Gamma^R + \mathbf{P}$, it is (by theorem 3) isomorphic to some pair-faithful $\mathcal{M}^+$, and we can find a pair-free $\mathcal{M}$ by cutting the pairs from the domain.

Since $\mathcal{M}^+$ is pair-faithful, by theorem 3 $\mathcal{M}^+ \models \mathbf{P}$, and it is clear that $\mathcal{M}, a \models Pr(x)$ iff $a(x)$ is a pair. Thus a simple induction shows that $\mathcal{M} \models \phi$ iff $\mathcal{M}^+ \models \phi^R$. ∴

4 The Translation

As a matter of preference, we will code up relations 'on the right': $\langle x, y, z \rangle$ will be understood as $\langle x, \langle y, z \rangle \rangle$. This makes it straightforward to treat arbitrary n-tuples in our finite pairing theory. There is some risk of ambiguity: after all, $\langle x, \langle y, x \rangle \rangle$ can be considered both a pair and a triple. To avoid confusion, we will say that t is an n-tuple if there is some resolution of the ambiguity according to which it is an n-tuple, and it is a strict n-tuple if it is an n-tuple and for no $m > n$ is it an m-tuple. Thus, if none of x, y, or z are tuples, $\langle x, \langle y, z \rangle \rangle$ is both a pair and a triple, but is only a strict triple.

We now want to define a translation from arbitrary second-order sentences to monadic second-order sentences with pairing. To do so, we'll want a way to say, of a monadic X, that all its elements are (not necessarily strict) n-tuples for a given n. To do this we'll give an inductive definition of some abbreviations.

Definition.

$$\begin{aligned}\overline{2}(p) &:= Pr(p)\\ \overline{n+1}(p) &:= Pr(p) \wedge \exists p'(\overline{n}(p') \wedge p' \prec_2 p)\\ \overline{1}(X) &:= X = X\\ \overline{n}(X) &:= \forall x(Xx \to \overline{n}(x))\end{aligned}$$

Of course, '$\overline{7}$' and the like are ambiguous between a first-order property of pairs and a second-order property of sets of pairs; context disambiguates. The oddity of the definition of $\overline{1}(X)$ is because we don't need to restrict translations of monadic second-order variables, but having a trivial restrictor makes the translation easier to define.

We also want a way to say that a given n-tuple counts as $\langle x_1, \ldots, x_n\rangle$. Here again, a recursive definition of a series of abbreviations is in order.

Definition.

$$\begin{aligned}p \approx_2 (x_1, x_2) &:= x_1 \prec_1 p \wedge x_2 \prec_2 p\\ p \approx_n (x_0, \vec{x}) &:= x_0 \prec_1 p \wedge \exists p'[p' \prec_2 p \wedge p' \approx_{n-1} (\vec{x})]\end{aligned}$$

Intuitively, $p \approx_n (\vec{x})$ says that p is the n-tuple $\langle \vec{x} \rangle$. The following can be shown by an easy induction:

Proposition 5.

If $\mathcal{M}$ is pair-normal, then $\mathcal{M}, a \models \overline{n}(p)$ iff $[\![p]\!]_a$ is an n-tuple, $\mathcal{M}, a \models \overline{n}(X)$ iff $[\![X]\!]_a$ is a set of n-tuples, and $\mathcal{M}, a \models p \approx_n \vec{\alpha}$ iff $[\![p]\!]_a = \langle \overrightarrow{[\![\alpha]\!]_a} \rangle$.

Since the second-order variables are countably infinite for each adicity, we can help ourselves to a one-to-one correspondence v between monadic predicate variables and the entire space of predicate variables. The translation makes use of this v. Note also that the translation treats predicate constants and monadic predicate variables different than other predicate variables.

Definition.

$$
\begin{aligned}
Tr(\Pi^n \vec{\alpha}) &:= \Pi^n \vec{\alpha} \\
Tr(\alpha = \beta) &:= \alpha = \beta \\
Tr(X^1 \alpha) &:= X^1 \alpha \\
Tr(X^{n+1} \vec{\alpha}) &:= \exists p (v(X^{n+1}) p \wedge p \approx_{n+1} (\vec{\alpha})) \\
Tr({\sim}\phi) &:= {\sim} Tr(\phi) \\
Tr(\phi \wedge \psi) &:= Tr(\phi) \wedge Tr(\psi) \\
Tr(\exists x \phi) &:= \exists x Tr(\phi) \\
Tr(\exists X^n \phi) &:= \exists v(X^n)[\overline{n}(v(X^n)) \wedge Tr(\phi)]
\end{aligned}
$$

5 The Equivalence

The main result here is

Theorem 6.

If $\mathcal{M}$ is pair-normal, then for any sentence ϕ, $\mathcal{M} \models \phi$ iff $\mathcal{M} \models Tr(\phi)$.

To prove the theorem we rely on a lemma. First, one more definition.

Definition.

Let $\mathcal{M}$ be a pair-normal model and V a set of variables. Then we say that a variable assignment is ***v*-well-behaved on *V*** iff

(i) for every first-order variable $x \in V$, $a(x)$ is not a pair; and

(ii) for every second-order variable $X^n \in V$, $a(v(X^n)) = \{\langle \vec{x} \rangle : \langle \vec{x} \rangle \in a(x)\}$.

(Recall that for any n, $v(X^n)$ is monadic.) Note that if needed, for a given formula ϕ, we can always re-shuffle variables to get an equivalent ϕ' where no variable Y in ϕ' is $v(X)$ for any X in ϕ'.

Since sentences, which have no free variables, are true relative to one assignment iff they are true relative to them all, theorem 6 is an immediate corollary of

Lemma 7.

If $\mathcal{M}$ is pair-normal, V contains all the variables occurring in ϕ, and a is v-well-behaved on V, then

$$\mathcal{M}, a \models \phi \text{ iff } \mathcal{M}, a \models Tr(\phi).$$

Proof.

By induction on the complexity of ϕ. The only non-trivial part of the base case is when ϕ is $X^n\vec{\alpha}$ and $n > 1$. In this case, let p be a first-order variable not free in ϕ. Then $\langle\overrightarrow{[\![\alpha]\!]_a}\rangle = a[\langle\overrightarrow{[\![\alpha]\!]_a}\rangle/p](p)$. Note that $\langle[\![\alpha]\!]_a\rangle$ is in D since $\mathcal{M}$ is pair-closed.

$$\begin{aligned}
\mathcal{M}, a \models X^n\vec{\alpha} &\text{ iff } \overrightarrow{[\![\alpha]\!]_a} \in a(X^n) && \text{(TAP)}\\
&\text{ iff } \langle\overrightarrow{[\![\alpha]\!]_a}\rangle \in \{\langle\vec{x}\rangle : \langle\vec{x}\rangle \in a(X)\} && \text{(equivalence)}\\
&\text{ iff } \langle\overrightarrow{[\![\alpha]\!]_a}\rangle \in a(v(X^n)) && (v\text{-well-behavior})\\
&\text{ iff } a[\langle\overrightarrow{[\![\alpha]\!]_a}\rangle/p](p) \in a(v(X^n)) && \text{(above)}\\
&\text{ iff } \mathcal{M}, a[\langle\overrightarrow{[\![\alpha]\!]_a}\rangle/p] \models v(X^n)p \wedge p \approx_n (\vec{\alpha}) && \text{(prop. 5)}\\
&\text{ iff } \mathcal{M}, a \models \exists p(v(X^n)p \wedge p \approx_n (\vec{\alpha})) && (\text{T}\exists)\\
&\text{ iff } \mathcal{M}, a \models Tr(X^n\vec{\alpha}) && (\text{Df. } Tr)
\end{aligned}$$

The truth-functional operators and first-order-quantifier are straightforward by the induction hypothesis, as is the second-order quantifier $\exists X^n$ when $n = 1$. When $n > 1$, note that if $S^n \subseteq D^n$, then if $t \in S^n$, $t \in D$ by repeated applications of pair-closure. Thus $S^n \subseteq D$. So $\mathcal{M}, a[S^n/Y^1] \models \overline{n}(Y^1)$ for any monadic Y^1. Furthermore, if $Y^1 = v(X^n)$, then $a[S^n/Y^1]$ is v-well-behaved, and so the induction hypothesis applies.

$$\begin{aligned}
\mathcal{M}, a \models \exists X^n\phi &\text{ iff for some } S^n \in D^n, a[S^n/X^n] \models \phi && (\text{T}\exists 2)\\
&\text{ iff } S^n \in D^n \text{ and } \mathcal{M}, a[S^n/X^n] \models Tr(\phi) && \text{(ind. hyp.)}\\
&\text{ iff } \mathcal{M}, a[S^n/v(X^n)] \models Tr(\phi) && \text{(above)}\\
&\text{ iff } \mathcal{M}, a[S^n/v(X^n)] \models \overline{n}(v(X^n)) \wedge Tr(\phi) && (*)\\
&\text{ iff } \mathcal{M}, a \models \exists v(X^n)(\overline{n}(v(X^n)) \wedge Tr(\phi)) && (\text{T}\exists 2)\\
&\text{ iff } \mathcal{M}, a \models Tr(\exists X^n\phi) && (\text{Df. } Tr)
\end{aligned}$$

To finish the argument we need to check $(*)$. In the left-to-right direction, it holds from our above observation that S^n satisfies $\overline{n}$. Conversely, if some $S \subseteq D$ satisfies $\overline{n}$, then it is a set of n-tuples and therefore also a member of D^n. $\therefore$

6 Discussion

Theorems 3 and 6, plus the mutual interpretability between second-order and modal logic, justify one version of the folklore. If we consider all the models of **P**, we see that a full second-order sentence is true on one of them if and only if its monadic translation is true on it, too. Thus the translations are equivalent on this restricted class of models. That's enough to at least convince us that there is *some* sort of mutual interpretability between second-order logic and monadic second-order logic with pairing.

But a residual worry lingers. What if there is some second-order theory that has countermodels, but none of its countermodels are also models of **P**? (There are in fact *are* some — for instance, ones that are inconsistent with **P**.) If that can happen, the claim of *equivalence* might seem too strong.

We might think this worry unfounded. After all, given well-known techniques for embedding pairing functions inside a rich enough theory, any infinite second-order model will contain an implicit pairing theory. So if $P(\P)$ is the conjunction of **P** and $\exists X^3(P(X^3))$ is the result of quantifying into the ¶-position of that conjunction, every infinite model will be a model of $\exists X^3(P(X^3))$. So then, so long as Γ doesn't contain any instances of ¶, we'll be able to take any model of Γ and extend it to a model of $\Gamma + \mathbf{P}$ by reinterpreting '¶' as some satisfier of $P(X^3)$.

This won't work for finite models, since there will be no guarantee that we have enough implicit pairs to go around. If we want a more general recipe for equivalence, we can appeal to the Field-conservativeness of **P** (theorem 4). So long as Γ doesn't contain ¶, it has a model if and only if $\Gamma^R + \mathbf{P}$ does. As a result, we can take the equivalence claim to be modulated by $(\cdot)^R$: a set of second-order sentences Γ has a model if and only if the monadic $Tr(\Gamma)^R + \mathbf{P}$ has a model. In this case, it is Tr^R, not Tr, that gives us the final equivalence between monadic second-order logic with pairing and full second-order logic. Thus the folklore is vindicated.[1]

References

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